No 11

THE

MATHEMATICAL GAZETTE.

LONDON:
GEORGE BELL & SONS, YORK ST., COVENT GARDEN,
AND BOMBAY.

SPHERICAL GEOMETRY.

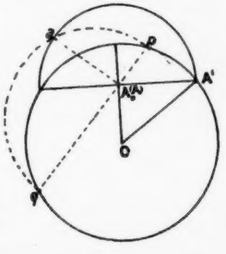
Papers read at the Annual General Meeting of the Association, March 20th.

The object of the papers is to consider two modes of representing points and lines on a spherical surface by points and lines on a plane: one method being by orthogonal projection, and the other by stereographic projection. The authors consider that students of spherical trigonometry ought to be able to accurately draw any figure with which they may have to deal, using one or other of the above methods of projection. The following propositions indicate methods of solving the various problems which would arise in connection with such accurate drawing.

I. ORTHOGONAL PROJECTION.

We will take the plane of the paper as the plane of projection, and for definiteness will take it through the centre of the sphere, thus dividing the sphere into two halves, one above and the other below the paper. The upper part may be called the visible hemisphere, and the bounding circle the horizon. The projections of points $A, B, C, \ldots$ on the sphere will be represented by points immediately below them, which we will call $A_0, B_0, C_0, \ldots$. In a diagram it would usually be sufficient to omit the suffixes.

(1) *To find the height of any point A above (or below) its representative point A_0.*



Let O be the centre of the sphere; then OA is a radius of the sphere, and the angle AA_0O is a right angle. Therefore, if A_0A' is drawn in the plane of the paper perpendicular to OA_0, to meet the horizon in A', $A'A_0$ is the required height. In fact, if the spherical surface be rotated about OA_0 though a right angle, A will come into the position A'.

During the rotation A will describe a circle perpendicular to the paper, with centre A_0 and radius A_0A'. This circle may, if desired, be depicted to scale *in* the plane of the paper, as shown.

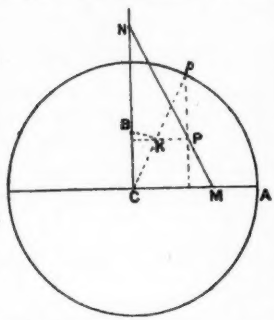
Generally, if rotation take place about *any* diameter of the sphere, A will trace a small circle whose projection on the paper is a straight line (pq)

E

through A_0, and such circle may be shown in plan by drawing a circle on pq as diameter. When A is vertically over A_0, its height above the plane is A_0a, where A_0a is drawn perpendicular to pq.

(2) All circles on the sphere will be represented by ellipses. If we can find the major and minor axes of these ellipses we can draw the ellipses themselves point by point, by means of a trammel or by any other convenient method. The major axes of the ellipses representing *great* circles are all diameters of the sphere, and the horizon circle is their common auxiliary circle. The major axes of ellipses representing small circles do not pass through the centre of the horizon circle, but their minor axes always do.

Trammel method.—If $MN=a+b$, and $NP=a$, $PM=b$, P will trace the ellipse whose semi-axes are $CA=a$, and $CB=b$, if MN slides with its ends on CM, CN.



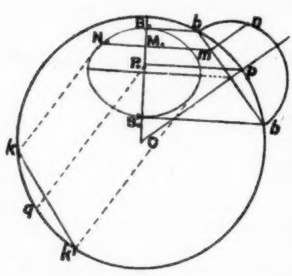
Corollary.—If we know the major axis, and a point on the ellipse, we can construct the trammel and find the length of the minor axis and draw the curve. For, if P be a point on the curve, we have only to take P as centre and a as radius, drawing an arc to cut CB in N, and produce NP to cut CA in M; then NM is the trammel, and $PM=b$.

Or, drawing Pp perpendicular to CA, joining Cp and cutting it in K by PK drawn parallel to CA, CK=length of semi-minor axis. [The connection between the two methods is simple; for Cp, MN are equally inclined to CM, since Cp, PN are equal hypothenuses of right-angled triangles having equal bases; hence $CK=MP$.]

(3) *To represent small circles with any point P as pole.*

(i.) Let P be on the horizon in the position p in the figure. The traces of the circles are chords, such as bb', of the horizon circle, drawn perpendicular to Op, and their angular distances from P are the arcs bp, ...

(ii.) Let P be any point on the sphere, represented by P_0. Join OP_0 and rotate the sphere round OP_0 through a right angle till P comes into the position p on the horizon.



Then bb' will be the trace of one of the required small circles whose diameter is bb'.

Rotate the system back again, then B_0B_0' will be the minor axis, both in magnitude and position, of the ellipse which represents the same circle. The length of the major axis will be bb' since that is the actual length of the diameter of the circle.

The ellipse can now be drawn in by the trammel method, or by the following method. The length of any chord perpendicular to OP_0 can be found thus:

Through any point M_0 on OP_0 draw M_0m perpendicular to OP_0 to meet bb' in m, and draw mn perpendicular to bb' to meet in n the semi-circle drawn on bb' as diameter; then mn is the length of the half chord (M_0N_0) along M_0m.

This can be made evident by imagining this semi-circle rotated round bb through a right angle, when n will be vertically over m.

Any number of tangents can be drawn to the ellipse from points on the horizon circle. For if the small circle be rotated in any direction till its pole comes to q on the horizon, its trace will be the line kk', where $qk=qk'=pb$; hence lines through k, k' drawn parallel to qP_0 must touch the ellipse.

(4) *To find the foci of an ellipse whose pole is represented by any point P_0.*

(i.) The foci F, F' lie on a circle, whose centre is O and radius OP_0.

For the semi-major axis $=Cb$, and the semi-minor axis $=Cb \,.\, \sin\theta$, where C is the middle point of bb', and θ is the angle COP_0;

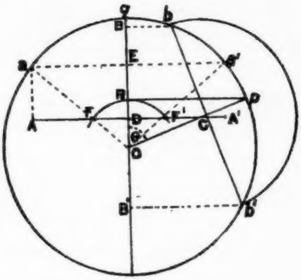
$\therefore\ DF=Cb \,.\, \cos\theta$,

and, from the figure,

$OD=OC \,.\, \cos\theta$;

$\therefore\ OF^2=OD^2+DF^2=(OC^2+Cb^2)\cos^2\theta$
$=r^2\cos^2\theta$;

$\therefore\ OF=r\cos\theta=OP_0$.



(ii.) If the small circle is rotated till P comes to q at the end of the line OP_0, the small circle will then be represented by the line aa' in the figure, and the foci will be on the lines Oa, Oa', and are therefore completely determined.

For $$\frac{FD}{OD}=\frac{Cb}{OC}, \text{ as proved above};$$

and Ea, EO are respectively equal to Cb, OC;

$$\therefore\ \frac{FD}{OD}=\frac{Ea}{EO}, \text{ which proves the proposition.}$$

Cor. i. The ellipses representing a series of copolar circles are all similar, and their common excentricity $=\cos\theta=OP_0\div r$.

Cor. ii. A line through F perpendicular to OC will go through the extremity (B') of the minor axis, for this line will make with FD the angle θ, whose cosine (by Cor. i.) $=FD\div FB'$.

Cor. iii. The following construction for the ellipse representing the great circle of which P is the pole, follows from the above theorem:

Draw the diameter of the horizon circle perpendicular to OP_0, and place foci F, F' on it by making $OF=OF'=OP_0$.

Find the ends B, B' of the minor axis by drawing a circle with F as centre, and r as radius, cutting OP_0 in B, B', the required points.

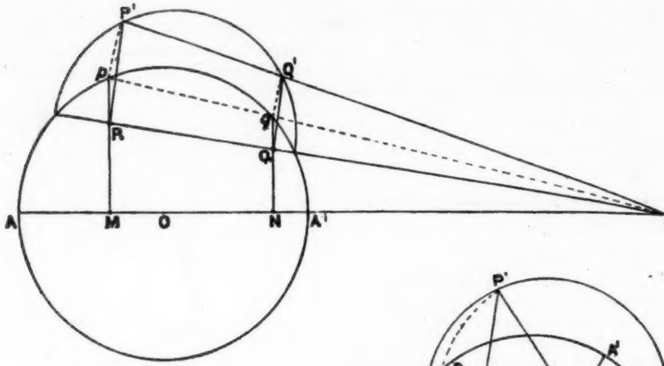
(5) *To find the major axis of the ellipse representing the great circle passing through two points P, Q.*

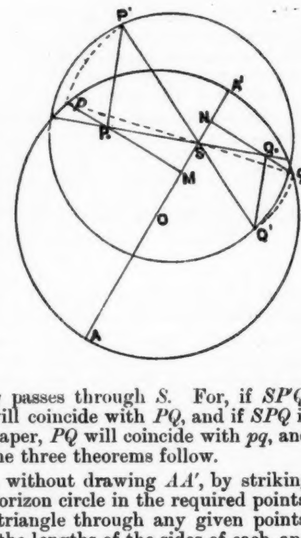
Join P_0Q_0 and draw P_0P', Q_0Q' perpendicular to P_0Q_0, and respectively equal to P_0P and Q_0Q in length, so that, if the figure $P_0Q_0Q'P'$ were rotated through a right angle round P_0Q_0, the points P', Q' would *coincide* with P, Q.

Let $P'Q'$ meet P_0Q_0 in the point S. Then evidently PQ meets P_0Q_0 in the same point.

[It may be noted that the same point S would be found in whatever direction P_0P', Q_0Q' are drawn, provided they are parallel to each other; and that S is a centre of similitude of spheres having P_0, Q_0 as centres, and P_0P, Q_0Q as radii. If P and Q are both above or both below the horizon, S is the external centre of similitude (Fig. 1), but if one is above, and the other below, S must be the internal centre of similitude (Fig. 2).]

Draw the line SO, cutting the horizon circle in A, A'; then AA' is the required major axis in magnitude and position. For it is the line of intersection of the great circle through P, Q with the horizon circle.





(6) *To find the true lengths of chord and arc PQ corresponding to two given points P_0, Q_0.*

In the above figures $P'Q'$ is evidently the required length of chord, and an arc drawn through $P'Q'$ with radius equal to that of the sphere would be the required length of arc.

Or, Having found S and AA' as above, draw P_0M, Q_0N perpendicular to AA', and produce them to meet the horizon circle in p, q. Then the chord and arc pq are the chord and arc required.

Riders. $Sp=SP'$ and $Sq=SQ'$ and pq passes through S. For, if $SP'Q'$ is rotated through 90° round P_0Q_0, $P'Q'$ will coincide with PQ, and if SPQ is rotated round AA' into the plane of the paper, PQ will coincide with pq, and S is on both axes of rotation. Whence the three theorems follow.

Cor. The points p, q can be obtained, without drawing AA', by striking arcs $P'p$, $Q'q$ with centre S, cutting the horizon circle in the required points.

It is now possible to draw a spherical triangle through any given points, and its polar triangle. Also we can find the lengths of the sides of each, and therefore the angles of each since the angles of one are the supplements of the sides of the other. However, the angles of a triangle can be found in another, and simpler, way.

(7) *To find the angle QPR of a spherical triangle PQR.*

Let the triangle revolve about the diameter of the sphere perpendicular to OP_0 till P comes vertically over O. Then the various points of the elliptic arcs P_0Q_0 and P_0R_0 describe straight lines parallel to P_0O, and the tangents to these arcs parallel to P_0O (*i.e.* the tangents at the ends of the diameters conjugate to OP_0) will be the paths of the points of those arcs which are farthest removed from P_0O; and the points on the sphere corresponding to these points will reach the horizon circle at the same time that P is vertically

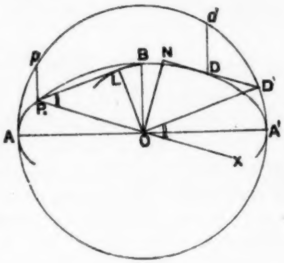
above O. If these points on the horizon circle are called D', E', the angle at P is equal to $D'OE'$. That is

If tangents be drawn to the ellipses P_0Q_0 and P_0R_0 parallel to P_0O to cut the horizon circle in D' and E', the angle at $P = D'OE'$.

But the best construction is based on a deduction from this, viz.:

The angle P is equal to the angle between tangents drawn from P_0 to the minor auxiliary circles of the ellipses P_0Q_0 and P_0R_0.

To prove this it is only necessary to draw one ellipse through P_0 as shown in the figure, and to show that the angle LP_0O = the angle $D'OX$, where P_0L is drawn to touch the minor auxiliary circle of the ellipse, and OX is the prolongation of P_0O. *Proof.*—Drawing ON perpendicular to the tangent DD', we have, by the properties of conjugate diameters,

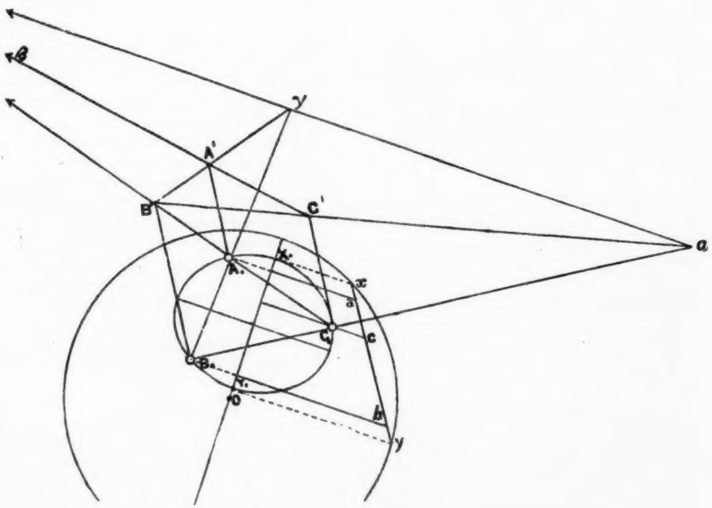


$$ON \cdot OP_0 = OB \cdot OA \text{ ; also } OL = OB \text{ and } OD' = OA \text{ ;}$$

$$\therefore OL \div OP_0 = ON \div OD', \therefore \text{angle } LP_0O = \text{angle } ND'O = \text{angle } D'OX. \quad \text{[Q.E.D.]}$$

(8) *To draw the ellipse corresponding to three given points ABC.*

Let A_0, B_0, C_0 be the projections of A, B, C; and find α, β, γ, the collinear points where BC, CA, AB meet the plane of the paper. (Prop. (5)).



Then the major axis of the required ellipse is parallel to $\alpha\beta\gamma$, and the minor axis is perpendicular to this line and passes through O.

The projection of ABC on a vertical plane through the minor axis is a straight line shown in plan by acb. If acb cuts the sphere in x, y, the length of the major axis is xy, and its projection X_0Y_0 is the minor axis. Hence the ellipse is completely determined.

Note.—The distances of a, b, and c from the minor axis X_0Y_0 are respectively equal to the heights $A'A_0$, $B'B_0$, and $C'C_0$. A. LODGE.

II. STEREOGRAPHIC PROJECTION.

It may be useful to compare with the results proved above, the interesting and comparatively little known properties of the Stereographic Projection of the sphere so far as they affect its practical application to spherical triangles. This projection, in which we take a point S on the sphere as the vertex of projection, and the central plane parallel to the tangent plane at S as the plane of projection, has certain great practical advantages, as shown in (1) and (2) below; and obviates the difficulty of drawing an ellipse.

(1) *All circles on the sphere project into circles on the plane and vice versa.*

Assuming the fundamental properties of cones of the second degree, that they have two sets of circular sections equally inclined to the axis, we have only to observe that, xy being the diameter in a plane through SO (the plane of the paper, Fig. 1) of any circle on the sphere and p its pole, Sp bisecting the angle xSy in this symmetrical plane is the axis of the cone of projection. For the planes of the circle xy and its projection DE, being parallel to the tangent planes at p and S, are equally inclined to Sp. But DE may also be proved to be a circle by elementary geometry; one proof that has been given consists in showing that, if xty be a tangent cone touching the sphere round the circle, the projections of all lines of the cone between t and the circle are equal, each line being to its projection as $St : SC$, where C is the projection of t. This has the advantage of giving a neat construction for the centre C. But probably the simplest proof of all is that founded on the proposition given on p. 93 of the *Gazette*, No. 136. The circle xy may be considered as the intersection of the given sphere with another; and we may so invert from centre S as to get for their inverses the plane of projection and a new sphere; whose intersection will be the projection of xy, and obviously a circle.

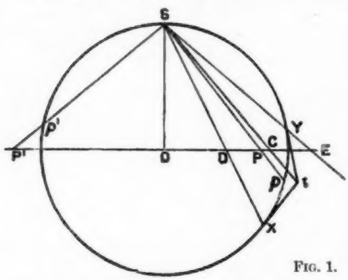


FIG. 1.

(2) *All angles on the sphere project into equal angles on the plane.*

Angles at p, for instance, on the sphere, being in a plane (the tangent plane at p) equally inclined to Sp with the plane of projection, and both at right angles to the symmetrical plane (that of the paper), the proposition follows at once. Thus, in stereographic projection, *there is no distortion of the figure in its smallest parts.*

(3) Each point in the projection corresponds to one, and only one, point on the sphere. Diametrically opposite points are not quite so simply related as

in orthogonal projection. If P, P' be the projections of such, PP' subtends a right angle at S, and therefore also at A, if OA be the radius at right angles to OP in the plane of projection, and $PO \,.\, OP' = OA^2$. Thus if P be given, P' is determined by making the angle PAP' a right angle.

(4) A great circle may be distinguished in the projection either (i.) by cutting the principal circle at the extremities of a diameter, or (ii.) by passing through any pair of "opposite" points.

(5) The system of great circles through any point p on the sphere will project into the co-axal system through P and P'. The system of circles with p as pole (since they cut the former series at right angles) will project into the co-axal system, with P, P' as limiting points. For shortness we may speak of P, P' in the projection as the "poles" of such a system.

(6) To find the pole of any given circle whose diameter passing through O is DE (Fig. 2), we have merely to take P the intersection with DE of the bisector of the angle DAE, where OA is the radius of the principal circle at right angles to DE. For since DP, PE subtend equal angles at A, they will subtend equal angles at S, and therefore correspond to equal arcs on the sphere lying on a great circle through S.

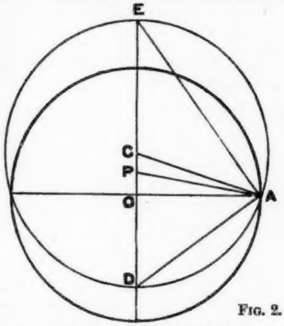


Fig. 2.

If the given circle be a great circle, A will be a point on it and DAE a right angle, as shown in the figure, and we have,

$$\text{angle } DAP = \text{angle } PAE = 45^\circ\,;$$

also,

$$DAO = AEC = CAE \text{ if } C \text{ be the centre.}$$

Therefore AP bisects the angle OAC. This shows how, for a great circle, we may at once find P if C be given, and C if P be given, and its radius is CA.

(7) It is now obvious how, given P, Q, R, to describe the triangle PQR, and also the polar triangle DEF (Fig. 3, next page). The opposite points P', Q', R' being found, the lines YZ, ZX, XY bisecting PP', QQ' RR' at right angles will intersect in X, Y, Z, the centres of the great circle arcs which form the sides of the triangle PQR.

And if points C_1, C_2, C_3 be taken on OP, OQ, OR produced such that the angles $PA_1C_1 = OA_1P$, $QA_2C_2 = OA_2Q$, $RA_3C_3 = OA_3R$ (where OA_1, OA_2, OA_3 are radii of the principal circle at right angles to OP, OQ, OR), the circular arcs described from these centres with radii C_1A_1, C_2A_2, C_3A_3 will make the polar triangle DEF.

Thus the two triangles may be described independently; but when the centres of the arcs QR, RP, PQ have been found their poles may be easily obtained, and these are the vertices of DEF; or we may first describe the polar triangle, and then its polar which will be PQR.

Again, we may otherwise describe a great-circle arc PQ, without finding P' and Q', by a method which, if less definite than the above, is practically shorter (Fig. 4). For if we describe *any* circle through P, Q to intersect the principal circle in E, F, the straight lines PQ, EF will intersect in K, the radical centre of these and the required circle. Then KO will be the radical axis of the principal and required circles since it must be a diameter, AB, of the former; and the centre of the latter will be determined as the intersection of the lines bisecting PQ, AB at right angles.

(8) When the triangle PQR is drawn, its angles will truly represent those of the original on the sphere. So, too, the sides of the latter will be given

by the exterior angles of the projected polar triangle. However, an arc equal to a side pq will be given thus (Fig. 4): Let tangents to PQ at P and Q intersect AB at L and M; then the circular arcs PU, QV described with

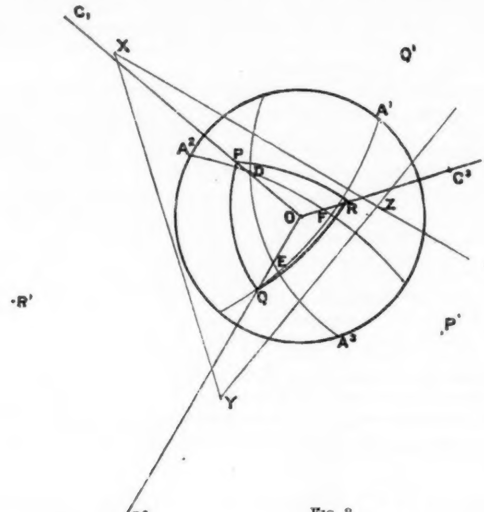


FIG. 3.

L, M as centres will be projections of small circles with A, B for poles, and, therefore, intercept an arc UV on the principal section equal to the original of PQ.

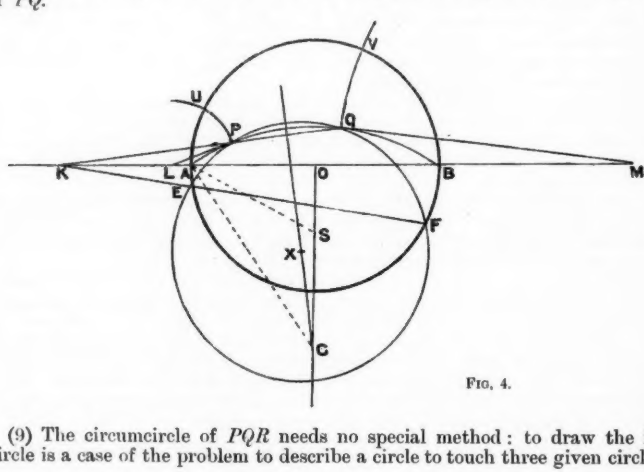


FIG. 4.

(9) The circumcircle of PQR needs no special method: to draw the in-circle is a case of the problem to describe a circle to touch three given circles.

To conclude by briefly enumerating some of the other more important constructions of spherical trigonometry, we observe that the projection of the middle point of pq may be determined, by taking the middle point of UV (Fig. 4), and describing through it the circle coaxal with PU, QV; the great-circle arc perpendicular to another at P is determined by having its centre on the tangent at P and on the line bisecting PP' at right angles; and in a similar way the arc having *any* given direction at a point, and in particular the bisector of a given spherical angle may be found. Such constructions may be applied to give the poles of the in-circle and circum-circle, more simply found, however, when these circles have been drawn. The drawing of a great-circle arc from P perpendicular to any other arc involves the description of a circle through two given points P, P' to cut a given circle orthogonally; its centre is the radical centre of the given circle and two point-circles P, P'.

P. J. HEAWOOD.

NOTES.

WITHIN a very few years of each other the two greatest English mathematicians of the century, Cayley and Sylvester, have passed away. Professor J. J. Sylvester died after a short illness on March 15th at the advanced age of eighty-three. No one has done so much to inspire enthusiasm for mathematical work in England, while he may be said to have created the present active spirit of research in America. His interest in our Association was shown by becoming an honorary member in 1886, and accepting the office of President for the year 1891.

An Association of Mathematical Teachers, under the presidency of Professor Rodolfo Bettazzi, has recently been formed in Italy on very similar lines to our own. Its title is the *Associazione Mathesis*.

An International Mathematical Congress is to be held on August 9, 10, 11 at Zurich in the Federal Polytechnic School. Mathematicians of all nationalities are cordially invited to attend.

In the "American Mathematical Monthly" for March we specially note an article by Dr. G. A. Miller *On the Solution of the Quadratic Equation*, and the 25th Proposition of Dr. G. B. Halsted's *Non-Euclidean Geometry: Historical and Expository*. This proposition is enunciated as follows:—*If two straight lines AX, BX, existing in the same plane (standing upon AB, one at an acute angle in the point A, and the other perpendicular at the point B) so always approach more to each other mutually, towards the parts of the point X, that nevertheless their distance is always greater than a certain assigned length, the hypothesis of acute angle is destroyed.*

The Cambridge University Press announce the following mathematical works in preparation:—Vol. XII. of "The Collected Mathematical Papers of the late Professor Cayley,"—Vol. II. of "The Scientific Papers of the late Professor Adams,"—"The Foundations of Geometry," by the Hon. B. Russell,—"A Treatise

on Abel's Theorem," by Mr. H. F. Baker,—"The Theory of Groups of a Finite Order," by Mr. Burnside,—"A Treatise on Universal Algebra, with some Applications," by Mr. A. N. Whitehead,—"A Treatise on Octonions," by Mr. A. M'Aulay,—"A Treatise on Spherical Astronomy," by Sir Robert Ball,—"A Treatise on Geometrical Optics," by Mr. R. A. Herman,—"An Elementary Course of Infinitesimal Calculus," by Professor Horace Lamb,—"Theoretical Mechanics," by Mr. A. E. H. Love,—and "The Works of Archimedes," edited in modern notation by Mr. T. L. Heath.

As announced on the cover, the Editor will gladly acknowledge by post all articles and notes sent to him; while all problems and solutions will either be inserted or acknowledged in the next number of the *Gazette* after they have been received. If any communications have not been acknowledged in the past, the Editor would be much obliged if the writers would kindly inform him of the fact.

MATHEMATICAL NOTES.

38. *On the proof of the formula* $s=ut+\frac{1}{2}at^2$.

Mr. E. T. Dixon has done well in drawing attention on p. 81 to the "fudge" by which most of the modern elementary text-books attempt to avoid the notion of limiting value in proving $s=ut+\frac{1}{2}at^2$.

One way to avoid the summation of infinitesimals would be to invert the order of proof.

If a body moves according to the law $s=ut+\frac{1}{2}at^2$, then the space is measured from the point at which the body is when $t=0$, and its velocity at any time t is $u+at$.

For
$$\frac{s_2-s_1}{t_2-t_1}=\frac{u(t_2-t_1)+\frac{1}{2}a(t_2{}^2-t_1{}^2)}{t_2-t_1}$$
$$=u+\tfrac{1}{2}a(t_2+t_1).$$

Taking the interval t_2-t_1 infinitesimal, we get in the limit, velocity at time $t=u+at$. But this is the velocity of a point which moves initially with velocity u, and is under a constant acceleration a.

Conversely, a body moving with this initial velocity and acceleration moves so that $s=ut+\frac{1}{2}at^2$; for its velocity is at every instant the same as that of the latter; hence its motion cannot be different.

In this connection, I should like to point out another favourite "fudge," by which elementary writers have tried to avoid the inevitable idea of limiting ratio.

"Variable Velocity is measured at any instant by the space which would be passed over in a second, if the velocity at the instant considered remained *the same throughout the second.*"

As is well pointed out in Garnett's *Elementary Dynamics*, § 7, "the words in italics take us back to the original difficulty, so that we appear to gain very little by the definition." In fact the postulated uniform velocity is to be the same as something not yet defined, and the supposed definition is a mere logical circle.

Of course if variable velocity be once properly defined, the above statement is a perfectly true proposition, requiring and admitting of proof.

R. F. Muirhead.

39. *Some Trigonometric Inequalities.*

In Dr. E. W. Hobson's *Treatise on Trigonometry*, pp. 86, 87, Exs. 4, 6, certain inequalities are given. These and some others are proved below.

(i.) *If the sum of two angles* $x+y$ *is given and equal to* $2a$, *then* $\sin x \,.\, \sin y$ *lies between* $\sin^2 a$ *and* $-\cos^2 a$.

From the formula $\sin^2 A - \sin^2 B = \sin(A+B) \,.\, \sin(A-B)$, we have

$$\sin x \,.\, \sin y = \sin^2\frac{x+y}{2} - \sin^2\frac{x-y}{2}$$

$$= \sin^2 a - \sin^2\frac{x-y}{2}, \text{ or } -\cos^2 a + \cos^2\frac{x-y}{2}.$$

Thus the maximum value of $\sin x \,.\, \sin y$ is $\sin^2 a$, and the minimum value is $-\cos^2 a$.

By increasing each of the angles x, y, a by $\frac{1}{2}\pi$, it follows that $\cos x \,.\, \cos y$ *lies between* $\cos^2 a$ *and* $-\sin^2 a$.

When either $\sin x \,.\, \sin y$ or $\cos x \,.\, \cos y$ is a maximum or a minimum, $\sin\dfrac{x-y}{2}$ or $\cos\dfrac{x-y}{2}$ must vanish, *i.e.* $x-y$ must be a multiple of π. This property can be extended to n angles x, y, ... whose sum is given and equal to na. In order that $\sin x \,.\, \sin y \ldots$ or $\cos x \,.\, \cos y \ldots$ may be a maximum or minimum, each and every pair of the angles must differ by a multiple of π; and the values of the products then are

$$(-1)^m \sin^n\left(a - \frac{m\pi}{n}\right) \text{ and } (-1)^m \cos^n\left(a - \frac{m\pi}{n}\right)$$

respectively, where m may be any integer. Without stopping to prove this general case, we will give some important particular cases.

(ii.) *If* x, y, ... *all lie between* 0 *and* π, *the maximum value of* $\sin x \,.\, \sin y \ldots$ *is* $\sin^n a$; *and if* x, y, ... *all lie between* $-\frac{1}{2}\pi$ *and* $+\frac{1}{2}\pi$ *the maximum value of* $\cos x \,.\, \cos y \ldots$ *is* $\cos^n a$.

For $\sin x$, $\sin y$,... and $\sin a$ are all positive; and since

$$\sin x \,.\, \sin y < \sin\tfrac{1}{2}(x+y) \,.\, \sin\tfrac{1}{2}(x+y),$$

it follows that so long as any two of the angles x, y, ... are unequal the product of their sines can be increased without altering the sum of the angles. Hence $\sin x \,.\, \sin y$, ... is a maximum when x, y, ... are all equal, *i.e.* the maximum value is $\sin^n a$.

Similarly for the cosines.

(iii.) *If* x, y, ... *all lie between* 0 *and* $\frac{1}{4}\pi$, *the maximum value of* $\tan x \,.\, \tan y \ldots$ *is* $\tan^n a$; *and the same is true if* x, y,... *are all positive, and their sum less than* $\frac{1}{2}\pi$.

$$\text{For } 1 - \tan x \,.\, \tan y = \frac{\cos(x+y)}{\cos x \,.\, \cos y}$$

$$> \frac{\cos(x+y)}{\cos^2\dfrac{x+y}{2}}, \text{ since } \cos(x+y) \text{ is positive,}$$

$$> 1 - \tan^2\frac{x+y}{2};$$

$$\therefore \quad \tan x \,.\, \tan y < \tan^2\frac{x+y}{2},$$

and the result may be extended to any number of angles in the same way as before.

If x, y, ... *all lie between* $\frac{1}{4}\pi$ *and* $\frac{1}{2}\pi$ *the minimum value of* $\tan x \,.\, \tan y \ldots$ *is* $\tan^n a$.

For if $x_1, y_1, \ldots$ and a_1 be the complements of $x, y, \ldots$ and a; then $x_1, y_1, \ldots$ lie between 0 and $\frac{1}{4}\pi$; therefore the maximum value of $\tan x_1 . \tan y_1, \ldots$ is $\tan^n a_1$, *i.e.* the maximum value of $\cot x . \cot y \ldots$ is $\cot^n a$, and the minimum value of $\tan x . \tan y \ldots$ is $\tan^n a$.

(iv.) *If $x, y, \ldots$ all lie between 0 and π, the maximum value of* $\sin x + \sin y + \ldots$ *is* $n \sin a$, *and the minimum value is* $\pm \sin na$.

If $x, y, \ldots$ all lie between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$, the maximum value of $\cos x + \cos y + \ldots$ *is* $n \cos a$, *and the minimum value is* $\pm \sin n(a + \frac{1}{2}\pi)$.

For $\sin x + \sin y = 2 \sin \frac{x+y}{2} . \cos \frac{x-y}{2} < 2 \sin \frac{x+y}{2}$.

Hence it follows that $\sin x + \sin y + \ldots$ is greatest when $x, y, \ldots$ are all equal, and its value is then $n \sin a$.

Again $(\sin x + \sin y)^2 - \sin^2(x+y)$,

$$= 4 \sin^2 \frac{x+y}{2} \left(\cos^2 \frac{x-y}{2} - \cos^2 \frac{x+y}{2} \right)$$

$$= 4 \sin^2 \frac{x+y}{2} . \sin x . \sin y, \text{ which is positive.}$$

Therefore if $x+y$ is constant, $(\sin x + \sin y)^2$ is least when $\sin x$ or $\sin y$ vanishes, that is when one of the angles x, y is 0 or π. Hence, since $x + y + \ldots = na$, it follows that $(\sin x + \sin y + \ldots)^2$ is least when all except one of the quantities $\sin x, \sin y, \ldots$ vanish, and its value is then $\sin^2 na$.

The theorem for the cosines can be deduced from that for the sines.

(v.) *If $x, y, \ldots$ all lie between 0 and $\frac{1}{2}\pi$, the minimum value of* $\tan x + \tan y + \ldots$ *is* $n \tan a$; *and if $x, y, \ldots$ are all positive, and their sum less than $\frac{1}{2}\pi$, the maximum value of* $\tan x + \tan y + \ldots$ *is* $\tan na$.

For
$$\tan x + \tan y = \frac{\sin(x+y)}{\cos x . \cos y}$$

$$> \frac{\sin(x+y)}{\cos^2 \frac{x+y}{2}}, \text{ from (i.),}$$

$$> 2 \tan \frac{x+y}{2},$$

and the result may be extended to any number of angles.

Also
$$\tan x + \tan y = \frac{2 \sin(x+y)}{\cos(x-y) + \cos(x+y)}.$$

Therefore if $x+y$ is constant, and less than $\frac{1}{2}\pi$, $\tan x + \tan y$ is a maximum when $\cos(x-y)$ is a minimum, that is when one of the angles x, y vanishes. Hence if $x, y, \ldots$ are all positive, and their sum na less than $\frac{1}{2}\pi$, $\tan x + \tan y + \ldots$ is a maximum when all but one of the angles $x, y, \ldots$ are zero, and its value then is $\tan na$.

E. Fenwick.

40. *To find the average Kinetic Energy of a harmonically vibrating particle during a complete period.*

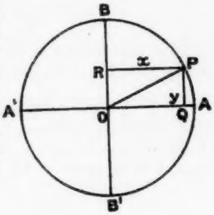
If, in the accompanying figure, a particle P traverses the circumference of the circle with uniform angular velocity ω, whilst another particle Q accompanies it, so as always to be at the foot of the perpendicular from P on the diameter AOA', Q executes harmonic vibrations in the period $\frac{2\pi}{\omega}$.

Also the velocity of Q is the component of the velocity of P along AA', and equal to $a\omega \sin AOP$, or $\omega . QP$.

Hence the kinetic energy of Q is $\frac{1}{2} m\omega^2 . QP^2$.

In finding the average kinetic energy of Q from A to A' and back, we have really to find the average value of QP^2, and multiply the result by $\frac{1}{2}m\omega^2$.

Suppose the particle starts from A. Divide the whole period in which the particle goes from A to A' and back into a large number $4n+4$ equal intervals of time; and let $P_1, P_2, \ldots P_n, B$ be the positions of the point P at the end of the 1^{st}, 2^{nd}, ... n^{th}, $(n+1)^{th}$ intervals respectively. Then, since P describes its circle uniformly, the arcs AP_1, $P_1P_2, \ldots P_{n-1}P_n, P_nB$ are all equal. Hence, if x_r, y_r are the perpendiculars from P_r to BB' and AA', the average value of QP^2 during the first quarter of the period is



$$\frac{y_1^2+y_2^2+\ldots+y_n^2+a^2}{n+1}=\frac{x_n^2+x_{n-1}^2+\ldots+x_1^2+a^2}{n+1}$$

$$=\frac{x_1^2+y_1^2+x_2^2+y_2^2+\ldots+x_n^2+y_n^2+2a^2}{2(n+1)}$$

$$=\frac{(n+2)a^2}{2(n+1)}=\frac{a^2}{2}, \text{ when } n \text{ is made indefinitely great.}$$

Hence the average value of the kinetic energy $\frac{1}{2}m\omega^2 \cdot QP^2$ during the first quarter of the period is $\frac{1}{4}m\omega^2a^2$, or half the maximum kinetic energy; and it is obviously the same for the whole period.

Corollary. For vibrations of a similar nature in the same medium, the period is the same for all particles, and therefore ω is constant.

Hence the average energies of the particles for such vibrations vary as the squares of their amplitudes. J. H. HERSCHKOWITZ.

41. *Extension of Euclid I. 47 to n-sided regular polygons.*

The following is a short summary of an article published in the *Indian Journal of Education* for December, 1895, showing how Euclid I. 47 can be extended to n-sided regular polygons without going beyond the first book.

Take equal lengths Aa, Bb, ... Ll, on the sides AB, BC, ... LA, of a given regular n-sided polygon; and upon them as bases describe outwards the isosceles triangles $AA'a$, $BB'b$, ... $LL'l$, having their base angles all equal to π/n. Then the regular polygon $A'B' \ldots L'$ is equal to the original one $AB \ldots L$ together with the n triangles described.

This is proved by showing that $A'a$ is equal and parallel to BB'; so that, if $A'B'$ cuts AB in a', the triangles $a'aA'$, $a'BB'$ are equal in all respects. Similarly, if $B'C'$ cuts BC in b', the triangles $b'bB'$, $b'CC'$ are equal, and so on. Now, if in the figure made up of $AB \ldots L$ and the n triangles $AA'a$, $BB'b$, ..., the triangles $a'BB'$, $b'CC'$, ... be replaced by $a'aA'$, $b'bB'$, ..., the new figure thus formed is $A'B' \ldots L'$, which proves the theorem.

Take a point O such that $A'O$ is bisected at right angles by AB. Then AOA' is an isosceles triangle, whose vertical angle OAA' is $2\pi/n$, equal to the triangle $AA'a$. Hence the n equal triangles $AA'a$, $BB'b$, ... are together equal to n times the triangle AOA', or to the n-sided regular polygon described on OA'. Also, AO is equal and parallel to $A'a$, and therefore equal and parallel to BB'; hence OB' is equal and parallel to AB, and at right angles to OA'. Hence we see that, in the right-angled triangle $A'OB'$, the n-sided regular polygon $A'B' \ldots L'$ described on $A'B'$ is equal to that described on OB' (the polygon $AB \ldots L$) together with that described on OA' (the n triangles $AA'a$, ... $LL'l$). Thus the extension of Euclid I. 47 to n-sided regular polygons is proved. V. RAMASWAMI AIYAR.

42. ***On the Equations to Chords and Tangents of Algebraic Curves.***

Let $\Sigma_m a_n x^m y^n=0$ be the equation to any curve; then the equation to the chord joining the points (x_1, y_1), (x_2, y_2) upon it is

$$\Sigma_m a_n\left\{(x-x_2)\cdot\frac{x_1^m-x_2^m}{x_1-x_2}\cdot y_2^n+(y-y_2)x_1^m\frac{y_1^n-y_2^n}{y_1-y_2}\right\}=0. \quad\ldots\ldots\ldots\ldots(1)$$

For this represents some straight line through the point (x_2, y_2); and, on putting $x=x_1$, $y=y_1$, the left-hand side becomes

$$\Sigma_m a_n\{x_1^m y_2^n-x_2^m y_2^n+x_1^m y_1^n-x_1^m y_2^n\},$$

which vanishes, because $\Sigma_m a_n x_1^m y_1^n=0$ and $\Sigma_m a_n x_2^m y_2^n=0$.

Putting $x_2=x_1$, $y_2=y_1$, after performing the divisions, we obtain the equation to the tangent at (x_1, y_1), viz.,

$$\Sigma_m a_n\{\overline{x-x_1}\,m x_1^{m-1}y_1^n+\overline{y-y_1}\,n x_1^m y_1^{n-1}\}=0,$$

or, $$x\Sigma_m a_n m x_1^{m-1}y_1^n+y\Sigma_m a_n n y_1^{n-1}x_1^m=\Sigma(m+n)_m a_n x_1^m y_1^n. \quad\ldots\ldots\ldots\ldots(2)$$

The above method is capable of extension to the case of homogeneous equations of any number of variables in the special cases in which not more than two variables occur in any one term, with the obvious simplification that the right-hand side of (2) vanishes because $(m+n)$ is common to all the terms.

Thus, the equation to a chord of $\alpha^3+k\beta^2\gamma=0$ is

$$(\alpha-\alpha_2)\frac{\alpha_1^3-\alpha_2^3}{\alpha_1-\alpha_2}+k(\beta-\beta_2)\frac{\beta_1^2-\beta_2^2}{\beta_1-\beta_2}\gamma_2+k(\gamma-\gamma_2)\beta_1^2\frac{\gamma_1-\gamma_2}{\gamma_1-\gamma_2}=0,$$

and therefore to a tangent,

$$(\alpha-\alpha_1)3\alpha_1^2+k(\beta-\beta_1)2\beta_1\gamma_1+k(\gamma-\gamma_1)\beta_1^2=0,$$

or, $$3\alpha_1^2\alpha+2k\beta_1\gamma_1\beta+k\beta_1^2\gamma=3\alpha_1^3+3k\beta_1^2\gamma_1=0.$$

In forming the equation to a tangent by this, or any other, method it is useful to observe that linear factors may be considered as merely replacing the α, β, γ, etc., of homogeneous coordinates. Thus, the equation to a tangent to $x^2+y^2=e^2(x\cos\alpha+y\sin\alpha-p)^2$ is

$$xx_1+yy_1=e^2(x\cos\alpha+y\sin\alpha-p)(x_1\cos\alpha+y_1\sin\alpha-p).$$

This tangent meets $x\cos\alpha+y\sin\alpha-p=0$ on the line $xx_1+yy_1=0$, which is perpendicular to $\dfrac{x}{x_1}=\dfrac{y}{y_1}$. The reader will recognize the conic property thus proved.

If the same method be employed with factors of any degree, we shall get a tangent curve. Thus, if we take the curve $(x^2)^2+(y^2)^2=1$, then

$$(x^2)(x_1)^2+(y^2)(y_1)^2=1$$

represents an ellipse touching at x_1, y_1, and of course at

$$(-x_1, y_1), (-x_1, -y_1), (x_1, -y_1).$$

R. W. GENESE.

43. *Suggested alternative proof of the results given in* § 246 *of Loney's "Analytical Conics"* (1*st edition*).

In the figure to § 243, draw SN, SM perpendicular to OX, OY respectively. Put $\lambda^2=a^2+b^2+2ab\cos\omega$, and designate the coordinates of S by x_1, y_1.

Then $$ON=x_1+y_1\cos\omega=ab(b+a\cos\omega)/\lambda^2,$$

$$OM=y_1+x_1\cos\omega=ab(a+b\cos\omega)/\lambda^2;$$

$\therefore$ the equation to MN, the tangent at the vertex, is

$$x/(b+a\cos\omega)+y/(a+b\cos\omega)=ab/\lambda^2.$$

Again, $$SO^2 = x_1^2 + y_1^2 + 2x_1y_1 \cos\omega = a^2b^2/\lambda^2.$$

Now the latus rectum $= 4SA = 4SM \sin SON$

$$= 4SM \,.\, SN/SO = 4a^2b^2 \sin^2\omega/\lambda^3.$$

If SA be produced to Z, so that $SA = AZ$,

then coordinates of Z are $\quad 2ab^2(b + a\cos\omega)^2/\lambda^4 - ab^2/\lambda^2,$

and $\quad 2a^2b(a + b\cos\omega)^2/\lambda^4 - a^2b/\lambda^2;$

hence, drawing a line through Z parallel to MN, we get, after a little reduction, the equation to the directrix. R. TUCKER.

EXAMINATION QUESTIONS AND PROBLEMS.

Our readers are earnestly asked to help in making this section of the GAZETTE *attractive by sending either original or selected problems.*

Solutions of problems should be sent within three months of the date of publication. They should be written only on one side of the paper, and without the use of contractions not intended for printing. Figures should be drawn with the greatest care on as small a scale as possible, and on a separate sheet.

181. If Σx^2 stand for $\quad x_1^2 + x_2^2 + \ldots + x_n^2,$

and Σyz for $\quad y_1z_1 + y_2z_2 + \ldots + y_nz_n,$

all the quantities being real, and if

$$\begin{vmatrix} \Sigma x^2, & \Sigma xy, & \Sigma xz \\ \Sigma yx, & \Sigma y^2, & \Sigma yz \\ \Sigma zx, & \Sigma zy, & \Sigma z^2 \end{vmatrix} = 0,$$

prove that
$$\left\Vert \begin{matrix} x_1, & x_2, & \ldots & x_n \\ y_1, & y_2, & \ldots & y_n \\ z_1, & z_2, & \ldots & z_n \end{matrix} \right\Vert = 0.$$

182. Find the maximum and minimum values of the ratio of the sum of the medians of a triangle to the sum of the sides.

183. If the diagonals of a quadrilateral are equal, the lengths of the perpendiculars to two opposite sides, measured from their point of intersection, are proportional to the sides.

184. Show how to inscribe a square in a given quadrilateral.

185. If a^2, b^2, c^2, x^2, y^2, z^2 are different from one another and from zero, and if the ratio of

$$\left(\frac{y}{z}+\frac{z}{y}\right)\left(\frac{z}{x}+\frac{x}{z}\right)\left(\frac{x}{y}+\frac{y}{x}\right) \text{ to } \left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{c}{a}+\frac{a}{c}\right)\left(\frac{a}{b}+\frac{b}{a}\right)$$

is unaltered when x and a are interchanged, prove that

$$\frac{x^2+y^2+z^2+a^2+b^2+c^2}{\dfrac{1}{x^2}+\dfrac{1}{y^2}+\dfrac{1}{z^2}+\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{c^2}} = \frac{b^2c^2y^2z^2}{a^2x^2}.$$

186. If $ABCD$ is a parallelogram, and EF, GH are any two lines parallel to AB, BC respectively, meeting AB in H, BC in F, CD in G, and DA in E; then AC, EG, HF are concurrent.

187. The difference of the sides, including a known angle, of a plane triangle being given, and also the sum of one of those sides and that opposite the given angle, to construct the triangle.
"Ladies' Diary," 1773.

188. In a given triangle to inscribe three circles touching each other and each touching one side of the triangle at its point of contact with the incircle. E. M. LANGLEY.

189. Find the area of the "nine-point nonagon" when the triangle is (i.) acute, (ii.) obtuse. W. J. GREENSTREET.

190. Given the trace of any curve $x=f(y)$, and its tangents, show how to construct geometrically the points and tangents of the curve $ax=yf(y)$, a being any constant. A. LODGE.

191. If OP and OQ be two forces acting at the point O, OR their resultant, and m and n their moments about a point A lying in their plane, prove that one of the anharmonic ratios of the pencil $O(APRQ)$ is equal to m/n. J. BRILL.

192. If the still air at A were transferred to B, β degrees farther north, it would cause a wind of b miles per hour; and if transferred to C, β degrees farther south, it would cause a wind in the contrary direction of c miles per hour; find the latitude of A when the ratio of b to c is 5 to 4 and β is equal to 20.
W. P. GOUDIE.

193. Show, without extracting the roots, that

$$\sqrt[3]{20-14\sqrt{2}}+\sqrt[3]{20+14\sqrt{2}}=4.$$ W. P. GOUDIE.

194. Prove that the complete solution of the equations

$$x\left(\frac{1}{y}+\frac{1}{z}\right)(y-z)^2=ax^3+bx^2+cx+d,$$

$$y\left(\frac{1}{z}+\frac{1}{x}\right)(z-x)^2=ay^3+by^2+cy+d,$$

$$z\left(\frac{1}{x}+\frac{1}{y}\right)(x-y)^2=az^3+bz^2+cz+d,$$

can be found without solving any equation of higher order than the third; and that the number of solutions, excluding those in which any zero or infinite roots occur, is 33. EDITOR.

195. Semi-circles are described outwards on the sides of a convex quadrilateral. Four particles start at the same instant, two from each end of one diagonal, and describe the semi-circles with uniform speeds in equal times. Prove that the lines joining

opposite particles at any time during the motion are equal, and that the difference of the rates of rotation of the lines is constant. EDITOR.

196. If a, b, c are three positive magnitudes such that b lies between a and c, and $a \sim b > b \sim c$, and $\frac{1}{b} \sim \frac{1}{a} > \frac{1}{c} \sim \frac{1}{b}$, then the same inequalities will hold with respect to A, B, C, where

$$A \equiv m + \frac{n}{a}, \quad B \equiv m + \frac{n}{b}, \quad C \equiv m + \frac{n}{c},$$

m and n being any positive magnitudes. R. F. MUIRHEAD.

The above is a correction of Problem 179, p. 90.

197. If $ABCD$ is a skew quadrilateral (that is a quadrilateral whose sides are not in one plane) with the angles A, B equal, and also the angles C, D equal, then $AD = BC$ and $AC = BD$.

R. F. MUIRHEAD.

198. If, in the quadrilateral of the preceding problem, E is the middle point of AB, and F that of CD, then EF is perpendicular both to AB and to CD. R. F. MUIRHEAD.

199. A has a counters and B has b counters; what is A's chance of winning all of B's counters, the play being even?

W. H. BESANT.

200. A number (n) of men, accompanied by a number (r) of tame monkeys, pick bananas, pile up the heap, and go to sleep. During the night one man gets up, gives each monkey a banana, divides the remainder of the heap into n equal portions, carries off and hides away one portion, piles up the heap again, and goes to sleep. A second man then wakes up, gives each monkey a banana, and, just as the first man did, takes away an n^{th} part of the remainder and goes to sleep. The rest of the men follow, do the same thing, and go to sleep. Finally, in the morning, all the men wake up, give each monkey a banana, and then find that the heap which is left can be divided into n equal portions. Find an expression for the least possible number of bananas; and prove that, for four men and one monkey, 1021 is the least possible number. W. H. BESANT.

SOLUTIONS.

133. *Pp is any chord of a conic, PG, pg the normals, G, g being on the axis; GK, gk are perpendiculars on Pp; prove that $PK = pk$.*

This is example 33, Chapter I., of Besant's *Geometrical Conics*, ninth edition, 1895, and, in the book of solutions of the examples, 1895, the solution is given as below. In the *Gazette*, p. 92, the question is only solved for central conics.

Let T be the intersection of the tangents at P and p, and draw TE perpendicular to Pp.

Then $$TE : PK :: TP : PG,$$

and $$TE : pk :: Tp : pg.$$

Again, draw GL, gl perpendicular to SP, Sp respectively, and TN, Tn perpendicular to SP, Sp respectively.

Then $$TP : PG :: TN : PL,$$

and $$Tp : pg :: Tn : pl.$$

But $TN = Tn$, since ST bisects the angle PSp, and $PL = pl$, for each is equal to the semi-latus rectum.

$$\therefore \quad TP : PG :: Tp : pg,$$

and $\therefore$ from above, $$TE : PK :: TE : pk,$$

$$\therefore \quad PK = pk.$$

145. *Show that the feet of the perpendiculars drawn from A to the internal and external bisectors of the angles B, C of the triangle ABC lie on the join of the mid-points of AB, AC.* G. Richardson.

Solution by C. F. Sandberg and W. E. Jeffares.

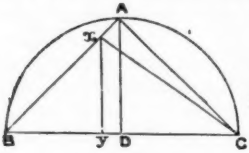
Let D, E be the mid-points of AB, AC, and AF, AG the perpendiculars from A to the internal and external bisectors of the angle B.

Then $AFBG$ is a rectangle, and its diagonals AB, GF are equal and bisect one another; therefore GF passes through D. Also since $DB = DF$, the angle ADF is double of DBF, or equal to DBC; therefore DF is parallel to BC, and GF passes through E. Hence F, G lie on DE. Similarly for the feet of the perpendiculars from A to the bisectors of the angle C.

146. *Prove geometrically that* $\tan^{-1} 13 - \cos^{-1}\frac{7}{\sqrt{85}} = \frac{\pi}{4}$. T. Roach.

Solution by Proposer and W. E. Jeffares.

Let AB, AC be two equal lines at right angles; draw AD perpendicular to BC; divide DB into 13 equal parts, and let DY be one of them; draw YX parallel to DA, meeting AB in X; join XC.



Then $13AX = AB = AC$, $\quad \therefore \; A\hat{X}C = \tan^{-1} 13$.

Also $DC = DB = 13DY$, $\quad \therefore \; CY = 14DY$;

and $YX = YB = 12DY$;

$$\therefore \quad X\hat{C}Y = \tan^{-1}\frac{12}{14} = \tan^{-1}\frac{6}{7} = \cos^{-1}\frac{7}{\sqrt{85}};$$

$$\therefore \quad \tan^{-1} 13 - \cos^{-1}\frac{7}{\sqrt{85}} = A\hat{X}C - X\hat{C}B = X\hat{B}C = \frac{\pi}{4}.$$

147. *Show that* $\cos^2(\theta+\gamma) + \cos^2(\theta-\gamma) - 2\cos(\theta+\gamma) \,.\, \cos(\theta-\gamma) \,.\, \cos 2\gamma$ *is independent of θ; and give a geometrical interpretation.* E. M. Langley.

Solution by C. F. Sandberg and W. E. Jeffares.

The given expression equals

$$\tfrac{1}{2}\{1 + \cos 2(\theta+\gamma)\} + \tfrac{1}{2}\{1 + \cos 2(\theta-\gamma)\} - (\cos 2\theta + \cos 2\gamma)\cos 2\gamma$$
$$= 1 + \cos 2\theta \,.\, \cos 2\gamma - \cos 2\theta \,.\, \cos 2\gamma - \cos^2 2\gamma$$
$$= 1 - \cos^2 2\gamma = \sin^2 2\gamma.$$

A geometrical interpretation may be given by taking a fixed circle, and drawing any two chords OP, OQ making an angle 2γ. Then if OM be the diameter through O, and θ the angle made by OM with the bisector of the angle POQ,

$$M\hat{O}Q=\theta+\gamma,\ M\hat{O}P=\theta-\gamma,\ OQ=OM\cos(\theta+\gamma),\ OP=OM\cos(\theta-\gamma);$$

$$\therefore\quad OM^2(\cos^2\overline{\theta+\gamma}+\cos^2\overline{\theta-\gamma}-2\cos\overline{\theta+\gamma}\,.\cos\overline{\theta-\gamma}\,.\cos 2\gamma)$$
$$=OQ^2+OP^2-2OQ\,.\,OP\cos P\hat{O}Q=PQ^2.$$

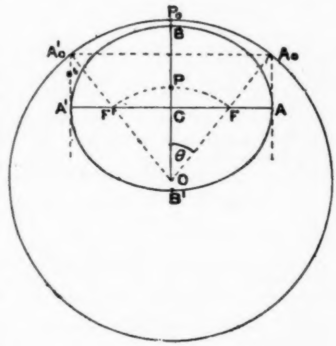
Hence $\qquad PQ^2=OM^2\sin^2 2\gamma=\text{constant};$

i.e. a chord of a circle which subtends a constant angle at any point on the circumference of a given circle is of constant length.

148. *If points on the surface of a sphere are represented by their orthogonal projections on the plane of the paper, prove that parallel circles on the sphere are represented by a set of similar ellipses, whose foci lie on a circle concentric with the sphere, and passing through the points representing the poles of the circles.* A. LODGE.

Solution by PROPOSER.

Let O be the centre of the sphere, whose radius$=r$. Let p be the pole of a circle whose angular radius is θ. Let P be the projection of p, and let ACA', BCB' be the major and minor axes of the ellipse representing the small circle.



Then $\qquad CA=r\sin\theta,$ ....(1)

If Op makes an angle a with the plane of the paper (which is the plane of projection), we see, by considering the section through OPp, that

$$\left.\begin{aligned} OP&=r\cos a,\\ \text{and}\quad OB&=r\cos(a-\theta),\\ OB'&=r\cos(a+\theta).\end{aligned}\right\}\text{.........(2)}$$

Hence

$CB=\frac{1}{2}(OB-OB')=r\sin a\sin\theta,$...(3)

and

$OC=\frac{1}{2}(OB+OB')=r\cos a\cos\theta,$...(4)

Let F be a focus of the ellipse, then

$$CF^2=CA^2-CB^2=r^2\cos^2a\sin^2\theta, \text{..............................(5)}$$

$$\therefore\quad OF^2=OC^2+CF^2=r^2\cos^2 a,$$

$$\therefore\quad OF=OP.$$

Also the eccentricity of the ellipse$=CF\div CA$

$$=\cos a=OP\div r.$$

Consequently the circles whose pole is p are represented by ellipses having a common eccentricity $OP\div r$, and having their foci F on a circle whose centre is O and radius OP. [Q.E.D.]

Further, since from (5) and (4) the ratio of CF to $OC=\tan\theta$, it follows that, if AA_0, $A'A_0'$ are drawn parallel to OP to meet the sphere in A_0, A_0', the foci of the ellipse whose major axis is AA' will lie on OA_0 and OA_0'

respectively; for the angle $A_0OP=\theta$. Hence the ellipse representing the circle whose angular radius is θ, is completely determined for any given position of P.

It should be noticed that when P is in the position P_0, the projection of the circle is the straight line A_0A_0'.

149. *In the process of converting* 1/100103 *into a repeating decimal, the* 273rd *remainder is* 100067, *the* 6126th *remainder is* 14, *and the* 7587th *remainder is* 100091. *Find where the remainders* 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 *will occur.*

C. N. MURTON.

Solution by PROPOSER.

The solution of the question depends on the following properties of repeating decimals:

Suppose that the fraction $\dfrac{1}{2m+1}$, when converted into a decimal, has a period of $2m$ figures. Then $2m+1$ must be a prime.

Also (i.) *the sum of the* p^{th} *and* $(m+p)^{th}$ *remainders is* $2m+1$.

For
$$\frac{1}{2m+1}=\cdot\dot{d}_1d_2\ldots\dot{d}_{2m}=\frac{d_1d_2\ldots d_{2m}}{10^{2m}-1},$$

$$\therefore\quad 2m+1 \text{ divides } 10^{2m}-1 \text{ or } (10^m-1)(10^m+1);$$

but $2m+1$ does not divide 10^m-1, otherwise the decimal would have m repeating figures or less; hence $2m+1$ divides 10^m+1.

Let R_p be the p^{th} remainder, then

$$\frac{10^p}{2m+1}=d_1d_2\ldots d_p+\frac{R_p}{2m+1},\quad \frac{10^{m+p}}{2m+1}=d_1\ldots d_{m+p}+\frac{R_{m+p}}{2m+1};$$

adding, we see that $2m+1$ divides R_p+R_{m+p};

$$\therefore\quad R_p+R_{m+p}=2m+1.$$

(ii.) Since
$$\frac{10^p}{2m+1}=d_1d_2\ldots d_p+\frac{R_p}{2m+1},$$

$$\therefore\quad \frac{10^p\,.\,R_p}{2m+1}=\text{integer}+\frac{R_p^2}{2m+1},$$

$\therefore$ p^{th} remainder in converting $\dfrac{R_p}{2m+1}$, viz. R_{2p}, is equal to R_p^2, provided R_p^2 is less than $2m+1$. Similarly $R_{3p}=R_p^3$, if R_p^3 is less than $2m+1$, etc.

$\frac{1}{100103}$ gives rise to a decimal having a repeating period of 100102 figures.

The 273rd remainder is $100067=100103-36$,

$\therefore$ the $(\frac{100102}{2}+273)^{th}$ or 50324th remainder is 36 or 6^2,

$\therefore$ 25162th remainder is 6. ....(1)

The 7587th remainder is $100091=100103-12$,

$\therefore$ $(50051+7587)^{th}$ or 57638th remainder is 12.

$\frac{1}{100103}$ gives 25162 decimal figures and remainder 6,

$\therefore$ $\frac{2}{100103}$ gives 25162 decimal figures and remainder 12,

$\therefore$ the remainder 2 occurs 25162 places before the 12,

$\therefore$ 2 is the $(57638-25162)^{th}$ or 32476th remainder, ...(2)

2^2 or 4 is the 64952th remainder, ...(3)

8 is the 97428th remainder, ...(4)

Again $\frac{1}{100103}$ gives 32476 decimal figures and remainder 2,

$\therefore$ $\frac{3}{100103}$ " 32476 " " " 6,

∴ remainder 3 occurs 32476 places before remainder 6, or what comes to the same thing $100102-32476$, *i.e.* 67626 places after it.

$\therefore$ 3 is the $(67626+25162)^{\text{th}}$ or 92788^{th} remainder, ...(5)

$\therefore$ 3^2 or 9 is the $(2\times 92788)^{\text{th}}$ or $(100102+85474)^{\text{th}}$ or 85474^{th} remainder. (6)

The 1^{st} remainder and also the 100103^{rd} is 10.

$\frac{1}{100103}$ gives 32476 decimal figures and remainder 2,

$\therefore$ $\frac{5}{100103}$ gives 32476 decimal figures and remainder 10,

$\therefore$ the remainder 5 occurs 32476 places before remainder 10,

$\therefore$ 5 is the 67627^{th} remainder. ....(7)

Similar reasoning starting from 14 as the 6126^{th} remainder will give 7 as the 73752^{th} remainder. ....(8)

151. *Prove that*

$$\sqrt{9\sqrt{6}+6\sqrt{12}}+\sqrt{9\sqrt{6}-6\sqrt{12}}=2\sqrt[4]{216}.$$

Solution by W. P. Goudie and W. J. Greenstreet.

Put $$x=\sqrt{9\sqrt{6}+6\sqrt{12}}+\sqrt{9\sqrt{6}-6\sqrt{12}};$$

then
$$\begin{aligned}x^2&=9\sqrt{6}+6\sqrt{12}+2\sqrt{486-432}+9\sqrt{6}-6\sqrt{12}\\&=18\sqrt{6}+6\sqrt{6}\\&=24\sqrt{6}=4\sqrt{216};\end{aligned}$$

$$\therefore\quad x=2\sqrt[4]{216}.$$

153. *A segment of a circle is placed with its base vertical, prove that the time from rest down any chord through the highest point is constant if the coefficient of friction has a certain fixed value.*

If AB is the vertical base, the angle of friction λ must be equal to the angle that the tangent at A makes with the horizon, or the angle that AB makes with the diameter of the circle through A. Taking this value for λ, it is easily proved that the time from rest down any chord through A is constant, and equal to the time of falling down AB.

154. *Prove that any weight up to* $\frac{1}{2}(3^n-1)$ *ounces can be determined correct to an ounce by using a pair of scales and* n *fixed weights.*

Every number up to $1+3+3^2+\ldots+3^{n-1}$, or $\frac{1}{2}(3^n-1)$, can be expressed in the form $a_0+3a_1+3^2a_2+\ldots+3^{n-1}a_{n-1}$, where $a_0, a_1, \ldots, a_{n-1}$ are restricted to the values 0, ± 1. To express any number in this form, we have only to bring it to the scale of 3, changing the remainder 2, whenever it occurs, to -1, and adding 1 to the corresponding quotient. Hence by using fixed weights of 1, 3, 3^2, ... and 3^{n-1} ounces, and putting some of them in one scale pan and some in the other, we can either determine the required weight of a body in ounces, if it be any whole number up to $\frac{1}{2}(3^n-1)$, or else two consecutive numbers between which it lies.

155. *Construct the circle which passes through a given point and cuts two given circles orthogonally.*

Solution by W. J. Greenstreet and W. E. Jeffares.

Let P be the given point, C, C' the given circles, L, M the middle points of tangents from P to C, L', M' the middle points of tangents from P to C', and O the point of intersection of LM and $L'M'$. Then the circle with centre O and radius OP is the one required. For LM is the radical axis of C and the point-circle P. Hence OP is equal to the tangent from O to C, and similarly also to the tangent from O to C'.

156. Mr. W. E. Jeffares has pointed out that this question is wrong. The only triangles of minimum perimeter circumscribed to a given triangle are triangles of zero perimeter, obtained by joining any point to the vertices of the given triangle.

158. *If squares be described outwards on the sides of a convex quadrilateral the line joining the centres of two opposite squares is equal and perpendicular to the line joining the centres of the other two.*

Solution by W. J. Greenstreet.

Let O_1, O_2, O_3, O_4 be the centres of the squares on AB, BC, CD, DA, and E, F, G, H, X the middle points of AB, BC, CD, DA, BD.

Then $EX=\frac{1}{2}AD=O_4H$; $HX=\frac{1}{2}AB=O_1E$.

Hence the triangles O_1EX and O_4HX have two sides equal and perpendicular.

$\therefore$ O_4X is equal and perpendicular to O_1X

Similarly O_3X.......................................O_2X.

$\therefore$ the triangles O_1XO_3 and O_2XO_4 have two sides equal and perpendicular.

$\therefore$ O_1O_3 and O_2O_4 are equal and perpendicular.

French geometers call such a quadrilateral as $O_1O_2O_3O_4$ a pseudo-square.

If D and C coincide we get a well-known property of the triangle: If squares are constructed externally on the sides of a triangle ABC, centres O_1, O_2, O_3, then AO_1, O_1O_2; BO_2, O_1O_3; CO_3, O_1O_2 are respectively equal and perpendicular.

It has also been noticed that X is the common centre of interior squares on O_1O_4, O_2O_3. So Y the mid point of AC will be the common centre of interior squares on O_1O_2, O_3O_4.

The quadrilateral $ABCD$ and pseudo-square $O_1O_2O_3O_4$ have the same centre of mean distances.

For a trigonometrical solution by Quint and Barisien v. *Mathesis*, vol. 5, 1895, p. 124.

159. *Prove that all triangles circumscribed to a rectangular hyperbola are obtuse-angled.*

Draw lines from the centre of the hyperbola in the directions of the sides of any circumscribed triangle taken in order. The angles between these lines are the supplements of the angles of the triangle. Each of the three lines lies in one of the external angles between the asymptotes; and two must lie in the same external angle, while the third lies in the vertically opposite. Hence the angle between the first two lines is acute, and the corresponding angle of the triangle is obtuse. The only exception is when the circumscribed triangle is formed by the two asymptotes and any third tangent.

An interesting way of stating the theorem is as follows: Although an equilateral hyperbola may in general be made to satisfy four conditions, yet it is possible to assign only three conditions which it cannot satisfy, viz., to touch the sides of any given acute-angled triangle.

173. *Between two sides of a triangle to inflect a straight line which shall have given ratios to the segments of the sides between it and the base.* (Proposed in 1864 by Mr. R. Tucker.)

The following solution is one of several given by Dr. J. S. Mackay in the *Proceedings of the Edinburgh Mathematical Society*, vol. iii., pp. 40, 41:

Let ABC be the triangle, and let $p:q$ and $r:q$ be the ratios of the segments of the sides to the inflected straight line.

From BA cut off $BD=p$; through D draw DE parallel to BC. Cut off $CF'=r$; with centre F' and radius$=q$, cut DE or DE produced at the points G'; and join $F'G'$. Let CG' meet AB or AB produced at G, and draw GF parallel to $G'F'$. GF is the line required.

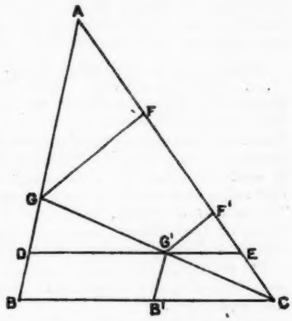
For through G' draw $G'B'$ parallel to GB.

Then $B'G'=BD=p$.

Now, since the quadrilaterals $CB'G'F'$, $CBGF$ are similar, and either similarly or oppositely situated, C being their centre of similitude; and since

$$B'G' : G'F' : F'C=p : q : r;$$

therefore $BG : GF : FC=p : q : r$.



We are also indebted to Dr. MACKAY for the following interesting historical note. It will show how fortuitously mathematical news is disseminated.

The problem was first proposed by Mr. Tucker in the *Educational Times* in 1864, and his solution of it, essentially the same as that given above, will be found in the *E. T. Reprint*, vol. ii., p. 16. Ten years later, it was proposed by Professor Neuberg in the *Nouvelle Correspondance Mathématique*, vol. i., p. 110, and solved by him in vol. ii., p. 248. Ten years later still, the particular case when p, q, r are all equal, was, at the instance of Mr. James Edward, brought before the January meeting of the Edinburgh Mathematical Society by Professor Chrystal, who gave a solution which has not been published. A solution by Mr. Edward is given in the Society's *Proceedings*, vol. ii., pp. 5-6, and another by myself in the same volume at p. 27. In the *Educational Times* for February, 1884, this particular case is proposed for solution by Mr. Edward, and in April of the same year the mathematical editor reproposes Mr. Tucker's problem, solutions of which appear in the number for October. In Vuibert's *Journal de Mathématiques Élémentaires* for 15th Nov., 1884, the particular case is proposed, and two solutions are given a month afterwards. In the *Proceedings* of the Edinburgh Mathematical Society, vol. iii., pp. 40-42, three solutions of the general problem, essentially the same, are given by myself, and a solution of the particular case by Mr. R. J. Dallas. Some years afterwards I found that the particular case went as far back as 1773-74. It occurs in the solution of a problem proposed in the *Ladies' Diary* for 1773 by Thomas Moss, which is reproduced above (p. 112). The solution will be given in the next number of the *Gazette*.

Solutions to all the Problems in No. 10, with the exception of 170-172 and 180, have been received. Many of these are in type, but there has not been room for their insertion in the present number. A solution of 178 is given in Prof. A. Lodge's paper (p. 100). Problem 179 was incorrectly worded, and has been restated in 196.

REVIEWS AND NOTICES.

The Elements of Algebra. By R. LACHLAN, Sc.D. (Edward Arnold.) The author tells us in the Preface that this forms the first part of a work on *Algebra*. We shall look forward with interest to the second part, for we have seen no better and more useful or reliable introduction to the subject. It would be distinctly to the advantage of school teaching if it became the standard text-book on elementary algebra. One of the most satisfactory features of the book is that nothing essential is omitted, and nothing unnecessary put in. Generally the best method of solving a question is given; not a useless variety. The book-work is very clear, concise, and complete. A considerable amount of deviation from the usual order is apparent; but nearly every change seems to us to result in improvement. We should however prefer to see some of the chapters divided into two or more, which could be effected with very little disturbance of the text. It might be well also

to place Simultaneous Quadratics after Division, on account of the difficulties which they present to beginners. F. S. M.

Solid Geometry. By PROFESSOR F. S. CAREY, M.A. (Sold by W. M. Murphy, Renshaw Street, Liverpool.) This is a small volume of 100 pages, in which the chief properties of the simpler solids are clearly explained and demonstrated. The elementary geometry of spherical triangles is dealt with very thoroughly; and we specially like the theorem in which the various cases of the identical equality of two spherical triangles are given in order in one long enunciation. F. S. M.

Mechanics of Fluids. By G. H. BRYAN, Sc.D., F.R.S., and F. ROSENBERG, M.A. (W. B. Clive.) This book is intended to prepare students for the Elementary Examination of the Science and Art Department on the Theoretical Mechanics of Fluids, and on the whole it is well adapted for the purpose. The first nine chapters form a condensed and somewhat indigestible introduction to Dynamics, and constitute the most unsatisfactory portion of the book. Surely it is well for even the elementary student to be taught that the principles of Dynamics rest upon the laws of motion, and are not to be proved by experiment. The rest of the book, dealing with the Mechanics of Fluids, is good. The principles of the subject, and experiments for their verification, are well explained, and clear descriptions are given of the usual instruments and apparatus.

The examples are numerous and well selected, and the ten examination papers on portions of the subject will serve to make the student pay some attention to the book-work. F. W. H.

BOOKS, MAGAZINES, ETC., RECEIVED.

Calculus for Engineers. By PROFESSOR J. PERRY, M.E., D.Sc., F.R.S. (Edward Arnold.)

First Stage Inorganic Chemistry. By G. H. BAILEY, D.Sc., and W. BRIGGS, M.A., F.R.A.S. (W. B. Clive.)

Professor J. J. Sylvester. A biographical article by Dr. G. B. HALSTED. ("Science.")

Fondamenti per una teoria generale dei gruppi. By PROF. R. BETTAZZI. (Tipografia Elzeviriana, Rome.) Also *Catena di un ente in un gruppo; Gruppi finiti ed infiniti di enti; Sulla definizione del gruppo finito.* ("Atti della R. Accademia delle Scienze di Torino," vols. xxxi., xxxii.)

Le superficie algebriche di genere lineare $p^{(1)}=2$, $p^{(1)}=3$. By PROF. FEDERIGO ENRIQUES. ("Rendiconti della R. Accademia dei Lincei," vol. vi.) Also *Sulle irrazionalità da cui può farsi dipendere la risoluzione d'un' equazione algebrica $f(x, y, z)=0$ con funzioni razionali di due parametri.* ("Math. Annalen," vol. xlix.)

The American Mathematical Monthly. January to April, 1897. Edited by PROF. B. F. FINKEL and J. M. COLAW, A.M. (A Hermann, 8 Rue de la Sorbonne, Paris.)

Journal de Mathématiques Élémentaires. March to May, 1897. Edited by PROF. M. DE LONGCHAMPS. (Librairie Ch. Delagrave, 15 Rue Soufflot, Paris.)

Periodico di Matematica. March to May, 1897. Edited by PROF. GIULIO LAZZERI. (Tipografia di Raffaello Giusti, Livorno.)

Bolletino della Associazione Mathesis. No. 3.

Bulletin de la Société Physico-Mathématique de Kasan. Vol. vi., Nos. 3, 4; and vol. vii., No. 1.

The following *Mathematical Notes* have been received:

On the connection between the inscribed and escribed circles of a plane triangle. By H. B. BILLUPS, M.A.

On the simplification of certain algebraical expressions. By E. M. LANGLEY, M.A.

On obtaining the integral solution of the indeterminate equation $y^2=ax+b$. By W. H. BESANT, Sc.D., F.R.S.

GLASGOW: PRINTED AT THE UNIVERSITY PRESS BY ROBERT MACLEHOSE AND CO.